BILINEAR FRACTAL INTERPOLATION AND BOX DIMENSION

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Abstract. In the context of general iterated function systems (IFSs), we introduce bilinear fractal interpolants as the fixed points of certain Read-Bajraktarević operators. By exhibiting a generalized "taxi-cab" metric, we show that the graph of a bilinear fractal interpolant is the attractor of an underlying contractive bilinear IFS. We present an explicit formula for the box-counting dimension of the graph of a bilinear fractal interpolant in the case of equally spaced data points.

Keywords and Phrases: Iterated function system (IFS), attractor, fractal interpolation, Read-Bajraktarević operator, bilinear mapping, bilinear IFS, box counting dimension.

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1. Introduction

Bilinear filtering or bilinear interpolation is used in computer graphics to compute intermediate values for a two-dimensional regular grid. One of the main objectives is the smoothening of textures when they are enlarged or reduced in size. In mathematical terms, the interpolation technique is based on finding a function $f(x, y)$ of the form $f(x, y) = a + bx + cy + dxy$, where $a, b, c, d \in \mathbb{R}$, that passes through prescribed data points.

As textures reveal, in general, a non-smooth or even fractal characteristics, a description in terms of fractal geometric methods seems reasonable. To this end, the classical bilinear approximation method is replaced by a bilinear fractal interpolation procedure. The latter allows for additional parameters, such as the box dimension, that are related to the regularity and appearance of an underlying texture pattern.

We introduce a class of fractal interpolants that are based on bilinear functions of the above form. We do this by considering a more general class of iterated function systems (IFSs) and by using a more general definition of attractor of an IFS. These more comprehensive concepts are primarily based on topological considerations. In this context, we extend and correct some known results from [12] concerning fractal interpolation functions that are fixed points of so-called Read-Bajraktarević operators. Theorem 4 relates the fixed point in Theorem 3 to the attractor of an IFS and generalizes known results to the case where the IFS is not contractive.

As a special example of the preceding theory we introduce bilinear fractal interpolants and show that their graphs are the attractors of an underlying contractive bilinear IFS. Finally, we present an explicit formula for the box dimension of the graph of a bilinear fractal interpolant in the case where the data points are equally spaced.

2. General iterated function systems

The terminology here for iterated function system, attractor, and contractive iterated function system is from [3]. Throughout this paper, $(\mathbb{X}, d)$ denotes a complete metric space with metric $d = d_{\mathbb{X}}$.

Definition 1. *Let $M \in \mathbb{N}$. If $f_m : \mathbb{X} \to \mathbb{X}$, $m = 1, 2, \ldots, M$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, ..., f_M)$ is called an **iterated function system** (IFS).*

By slight abuse of terminology we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and for the following mappings. We define $\mathcal{F} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by

$$\mathcal{F}(B) := \bigcup_{f \in \mathcal{F}} f(B)$$

for all $B \in 2^{\mathbb{X}}$, the set of subsets of $\mathbb{X}$. Let $\mathbb{H} = \mathbb{H}(\mathbb{X})$ be the set of nonempty compact subsets of $\mathbb{X}$. Since $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat $\mathcal{F}$ as a mapping $\mathcal{F} : \mathbb{H} \to \mathbb{H}$. When $U \subset \mathbb{X}$ is nonempty, we may write $\mathbb{H}(U) = \mathbb{H}(\mathbb{X}) \cap 2^U$. We denote by $|\mathcal{F}|$ the number of distinct mappings in $\mathcal{F}$.

Let $d_{\mathbb{H}}$ denote the Hausdorff metric on $\mathbb{H}$, defined in terms of $d_{\mathbb{X}}$. A convenient definition (see for example [5, p.66]) is

$$d_{\mathbb{H}}(B, C) := \inf\{r > 0 : B \subset C + r, C \subset B + r\},$$

for all $B, C \in \mathbb{H}$. For $S \subset \mathbb{X}$ and $r > 0$, $S + r$ denotes the set $\{y \in \mathbb{X} : \exists x \in S \ s.t.\ d_{\mathbb{X}}(x, y) < r\}$.

We say that a metric space $\mathbb{X}$ is *locally compact* to mean that if $C \subset \mathbb{X}$ is compact and r is a positive real number then $\overline{C + r}$ is compact. The notation $\overline{C + r}$ means the closure of the union of balls of radius r, one centered on each point of C.

The following information is foundational.

Theorem 1. *(i) The metric space $(\mathbb{H}, d_{\mathbb{H}})$ is complete.*

(ii) If $(\mathbb{X}, d_{\mathbb{X}})$ is compact then $(\mathbb{H}, d_{\mathbb{H}})$ is compact.

(iii) If $(\mathbb{X}, d_{\mathbb{X}})$ is locally compact then $(\mathbb{H}, d_{\mathbb{H}})$ is locally compact.

(iv) If $\mathbb{X}$ is locally compact, or if each $f \in \mathcal{F}$ is uniformly continuous, then $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is continuous.

(v) If $f : \mathbb{X} \to \mathbb{X}$ is a contraction mapping for each $f \in \mathcal{F}$, then $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is a contraction mapping.

Proof. (i) This is well-known. A short proof can be found in [5, p.67, Theorem 2.4.4].

(ii) This is well-known; see for example [9]. Here is a short proof. Let $\varepsilon > 0$ be given. Since $\mathbb{X}$ is compact we can find a finite set of points $\mathbb{X}_\varepsilon \subset \mathbb{X}$ such that $\mathbb{X} = \cup_{x \in \mathbb{X}_\varepsilon} \mathcal{B}(x, \varepsilon)$ where $\mathcal{B}(x, \varepsilon) \subset \mathbb{X}$ denote the open ball with center at x and radius ε. Let $\mathbb{H}_\varepsilon := 2^{\mathbb{X}_\varepsilon}$, a finite set of points in $\mathbb{H}$. It is readily verified that $\mathbb{H} = \cup_{C \in \mathbb{H}_\varepsilon} \mathcal{B}(C, \varepsilon)$ where now $\mathcal{B}(C, \varepsilon) \subset \mathbb{H}$ denotes the open ball with center at $C \in \mathbb{H}$ and radius ε, measured using the Haudorff metric. It follows that $\mathbb{H}$ is totally bounded. It follows that $\mathbb{H}$ is compact.

(iii) Let $C \in \mathbb{H}$. Consider the set $\overline{C + r}$. It belongs to $\mathbb{H}$ since $\mathbb{X}$ is locally compact. Let $\varepsilon > 0$ be given. Since $\overline{C + r}$ is a compact subset of $\mathbb{X}$ we can find a finite set of points $C_\varepsilon \subset \overline{C + r}$ such that $\overline{C + r} \subset \cup_{c \in C_\varepsilon} \mathcal{B}(c, \varepsilon)$. Let $\mathbb{C}_\varepsilon := 2^{C_\varepsilon}$, a finite set of points in $\mathbb{H}$. It is readily verified that $\overline{C + r} \subset \cup_{c \in \mathbb{C}_c} \mathcal{B}(C, \varepsilon)$ where now $\mathcal{B}(c, \varepsilon) \subset \mathbb{H}$ denotes the open ball with center at $c \in \mathbb{H}$ and radius ε, measured

using the Haudorff metric. It follows that $\overline{C+r}$ is totally bounded. It follows that $\overline{C+r}$ is compact.

(iv) Let $B \in \mathbb{H}$. We show that $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is continuous at B. We restrict attention to the action of $\mathcal{F}$ on $B+1$. If $\mathbb{X}$ is locally compact, it follows that $\overline{B+1}$ is compact. It follows that each $f \in \mathcal{F}$ is uniformly continuous on $\overline{B+1}$. It follows that if $\mathbb{X}$ is locally compact, or if each $f \in \mathcal{F}$ is uniformly continuous, we can find $\delta_\varepsilon > 0$ such that $d_{\mathbb{X}}(f(x), f(y)) < \varepsilon$ whenever $d_{\mathbb{X}}(x, y) < \delta_\varepsilon$, for all $x, y \in \overline{B+1}$, for all $f \in \mathcal{F}$. Let $C \in \mathbb{H}$ with $d_{\mathbb{H}}(B, C) < \delta_\varepsilon$ and let $f \in \mathcal{F}$. We can suppose that $\delta_\varepsilon < 1$.

Let $b' \in f(B)$. Then there is $b \in B$ such that $f(b) = b'$. Since $d_{\mathbb{H}}(B, C) < \delta_\varepsilon$ there is $c \in C$ such that $d(b, c) < \delta_\varepsilon$. Since $\delta_\varepsilon < 1$ we have $c \in B+1$. It follows that $d(f(b), f(c)) < \varepsilon$. It follows that $f(B) \subset f(C) + \varepsilon$. By a similar argument $f(C) \subset f(B) + \varepsilon$. Hence $d_{\mathbb{H}}(f(B), f(C)) < \varepsilon$.

(v) This is Hutchinson's theorem, [10, p. 731], proved as follows. Verify that if $\lambda \geq 0$ is a uniform Lipschitz constant for all $f \in \mathcal{F}$, namely $d_{\mathbb{X}}(f(x), f(y)) \leq \lambda d_{\mathbb{X}}(x, y)$ for all $f \in \mathcal{F}$, for all $x, y \in \mathbb{X}$, then λ is also a Lipschitz constant for $\mathcal{F}$, namely $d_{\mathbb{H}}(\mathcal{F}(B), \mathcal{F}(C)) \leq \lambda d_{\mathbb{H}}(B, C)$ for all $B, C \in \mathbb{H}$. If $f : \mathbb{X} \to \mathbb{X}$ is a contraction mapping for each $f \in \mathcal{F}$, then we can choose $\lambda < 1$. It follows that $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is a contraction mapping. ☐

For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k-fold composition of $\mathcal{F}$, i.e., the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k. Define $\mathcal{F}^0(B) := B$.

Definition 2. *A nonempty compact set $A \subset \mathbb{X}$ is said to be an* ***attractor*** *of the IFS $\mathcal{F}$ if*

(i) $\mathcal{F}(A) = A$ and

(ii) there is an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k\to\infty} \mathcal{F}^k(B) = A$, for all $B \in \mathbb{H}(U)$, where the limit is with respect to the Hausdorff metric.

The largest open set U such that (ii) is true is called the ***basin of attraction*** *(for the attractor A of the IFS $\mathcal{F}$).*

We will use the following observation [11, Proposition 3 (vii)], [5, p.68, Proposition 2.4.7].

Lemma 1. *Let $\{B_k\}_{k=1}^\infty$ be a sequence of nonempty compact sets such that $B_{k+1} \subset B_k$, for all k. Then $\cap_{k\geq 1} B_k = \lim_{k\to\infty} B_k$ where convergence is with respect to the Haudorff metric.*

We use the notation $\overline{S}$ to denote the closure of a set S.

Theorem 2. *Let $\mathcal{F}$ be an IFS with attractor A and basin of attraction U. If $\mathcal{F} : \mathbb{H}(U) \to \mathbb{H}(U)$ is continuous then*

$$A = \bigcap_{K\geq 1} \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)} \quad \text{for all } B \subset U \text{ such that } \overline{B} \in \mathbb{H}(U).$$

The quantity on the right-hand side here is sometimes called the *topological upper limit* of the sequence $\left\{F^k(B)\right\}_{k=1}^\infty$.

Proof. We carry out the proof under the assumption that $B \in \mathbb{H}(U)$. (It then follows from [11, Proposition 3 (i)] that Theorem 2 is true for all $B \subset U$ such that $\overline{B} \in \mathbb{H}(U)$.)

First we show that $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}$ is a nonempty compact subset of U. It is nonempty because it contains $\mathcal{F}^k(B)$ which is nonempty because $\overline{B}\in\mathbb{H}(U)$. Let $\{\mathcal{O}_i : i\in\mathcal{I}\}$ be an open cover of $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}$. Then $\{\mathcal{O}_i : i\in\mathcal{I}\}$ is an open cover of A. Hence it contains a finite subcollection, say $\{\mathcal{O}_m : m=1,2,..,M\}$ such that $A\subset\cup_{m=1}^M\mathcal{O}_m$. Since $\mathcal{F}^k(B)$ converges in the Hausdorff metric to A it follows that, for sufficiently large K, we have $\mathcal{F}^k(B)\subset\cup_{m=1}^M\mathcal{O}_m$ for all $k>K_1$. It follows that $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}\subset\cup_{m=1}^M\mathcal{O}_m$ for all $K>K_1$. It follows that, if $K>K_1$ then $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}$ is compact. If $K\leq K_1$ then we note that $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}=\bigcup_{K_1\geq k\geq K}\mathcal{F}^k(B)\cup\overline{\bigcup_{k>K_1}\mathcal{F}^k(B)}$ which is a finite union of compact sets, and so is compact. We conclude that $\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}$ is a nonempty compact set. It is also a subset of U since each term in the union belongs to U, so the union does belongs to U, so the closure of the union belongs to U.

Since $\cap_k\overline{\cup_{k\geq K+1}\mathcal{F}^k(B)}\subset\cap_k\overline{\cup_{k\geq K}\mathcal{F}^k(B)}$ it follows that

$$\tilde{A}=\bigcap_{K\geq 1}\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}$$

is a well-defined nonempty compact subset of U.

Now observe that

$$\begin{aligned}
\mathcal{F}\left(\tilde{A}\right) &= \mathcal{F}\left(\lim_{K\to\infty}\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}\right) \text{ (by Lemma 1)}\\
&= \lim_{K\to\infty}\mathcal{F}\left(\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}\right) \text{ (since } \mathcal{F}:\mathbb{H}\to\mathbb{H} \text{ is continuous)}\\
&= \lim_{K\to\infty}\bigcup_{f\in\mathcal{F}} f\left(\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}\right) \text{ (by the definition of } \mathcal{F})\\
&= \lim_{K\to\infty}\bigcup_{f\in\mathcal{F}}\overline{f\left(\bigcup_{k\geq K}\mathcal{F}^k(B)\right)} \text{ (since } f:\mathbb{X}\to\mathbb{X} \text{ is cts.)}\\
&= \lim_{K\to\infty}\overline{\bigcup_{f\in\mathcal{F}} f\left(\bigcup_{k\geq K}\mathcal{F}^k(B)\right)} \text{ (since } \bigcup_{f\in\mathcal{F}} \text{ is a finite union)}\\
&= \lim_{K\to\infty}\overline{\bigcup_{k\geq K}\bigcup_{f\in\mathcal{F}} f(\mathcal{F}^k(B))} \text{ (by direct comparison, see below,)}\\
&= \lim_{K\to\infty}\overline{\bigcup_{k\geq K}\mathcal{F}^k(B)}=\tilde{A}
\end{aligned}$$

(It is readily verified that $f\left(\cup_{k\geq K}\mathcal{F}^k(B)\right)=\cup_{k\geq K}f(\mathcal{F}^k(B))$ which implies the penultimate step above.)

Since $\tilde{A}$ is nonempty, compact and lies in U, the basin of attraction of A, we must have

$$A=\lim_{k\to\infty}\mathcal{F}^k(\tilde{A})=\tilde{A}. \qquad \square$$

We will also need the following observation.

Lemma 2. *Let $\mathbb{X}$ be locally compact. Let $\mathcal{F} = (\mathbb{X}; f_1, f_2, ..., f_N)$ be an IFS with attractor A and basin of attraction U. For any given $\varepsilon > 0$ there is an integer L such that for each $x \in A + \varepsilon$ there is an integer $l \leq L$ such that*

$$d_H(A, \mathcal{F}^l(\{x\}) < \varepsilon.$$

Proof. For each $x \in \overline{A + \varepsilon}$ there is an integer $l(x, \varepsilon)$ so that $d_H(A, \mathcal{F}^{l(x,\varepsilon)}(\{x\}) < \varepsilon/2$.

Since $\mathbb{X}$ is locally compact it follows that $\mathcal{F}^{l(x,\varepsilon)} : \mathbb{H} \to \mathbb{H}$ is continuous. Since $\mathcal{F}^{l(x,\varepsilon)} : \mathbb{H} \to \mathbb{H}$ is continuous there is an open neighborhood $N(\{x\})$ (in $\mathbb{H}$) of $\{x\}$ such that $d_H(A, \mathcal{F}^{l(x,\varepsilon)}(Y) < \varepsilon$ for all $Y \in N(\{x\})$. It follows, in particular, that there is an open neighborhood $N(x)$ (in $\mathbb{X}$) of x such that $d_H(A, \mathcal{F}^{l(x,\varepsilon)}(\{y\}) < \varepsilon$ for all $y \in N(x)$. Also since $\mathbb{X}$ is locally compact, there is a finite set of points $\{x_1, x_2, ..., x_q\}$ such that $\overline{A + \varepsilon} \subset \cup_{i=1}^{q} N(x_i)$. Choose $L := \max_i l(x_i, \varepsilon)$. □

3. Fractal interpolants as fixed points of operators

Let $\{(X_j, Y_j) : j = 0, 1, ..., N\}$ denote the cartesian coordinates of a finite set of points in the Euclidean plane, with

$$X_0 < X_1 < ... < X_N.$$

Let I denote the closed interval $[X_0, X_N]$. For $n = 1, 2, ..., N$, let $l_n : I \to [X_{n-1}, X_n]$ be a continuous bijection. Let $L : I \to I$ be bounded and such that

$$L(x) := l_n^{-1}(x) \text{ for } x \in (X_{n-1}, X_n)$$

for $n = 1, 2, ..., N$. Let $S : [X_0, X_N] \to \mathbb{R}$ be bounded and piecewise continuous, where the only possible discontinuities occur at the points in $\{X_1, X_2, ..., X_{N-1}\}$. Let

$$s := \max\{|S(x)| : x \in [X_0, X_N]\}.$$

Denote by $C = C(I)$ denote the set of continuous functions $f : I \to \mathbb{R}$. It is well-known that (C, d_∞) is a complete metric space, where

$$d_\infty(f, g) = \max\{|f(x) - g(x)| : x \in I\}.$$

Let

$$C^* := \{f \in C : f(X_0) = Y_0, f(X_N) = Y_N\},$$
$$C^{**} := \{f \in C : f(X_j) = Y_j \text{ for } j = 0, 1, ..., N\}.$$

Note that C^* and C^{**} are closed subspaces of C, with $C^{**} \subset C^* \subset C$. We say that each of the functions in C^{**} *interpolates the data* $\{(X_j, Y_j)\}$.

Let $b \in C^*$ and $h \in C^{**}$. Define $T : C \to C$ by

$$T(g) := h + S \cdot (g \circ L - b \circ L).$$

T is a form of Read-Bajraktarević operator as defined in [12]. The following result is a corrected version of [12, Theorem 5.1, p. 136]. See also [10, Theorem 3, p. 731].

Theorem 3. *The mapping $T : C \to C$ obeys*

$$d_\infty(Tg, Th) \leq s d_\infty(g, h)$$

for all $g, h \in C$. In particular, if $s < 1$ then T is a contraction and it possesses a unique fixed point $f \in C^{**}$.

Proof. To prove the inequality observe that

$$\begin{aligned} d_\infty(T(g),T(h)) &= \max\{|S(x)(g(L(x))-h(L(x)))| : x\in I\} \\ &\le s\max\{\left|(g(l_n^{-1}(x))-h(l_n^{-1}(x)))\right| : x\in[X_{n-1},X_n], n=1,2,...,N\} \\ &= sd_\infty(g,h). \end{aligned}$$

The existence of a unique fixed point $f\in C$ (when $s<1$) follows from the contraction mapping theorem. Since $f(C^*)\subset C^{**}$ and (C^{**},d_∞) is closed, hence complete, it follows that $f\in C^{**}$. □

Note that $Tg=H+S\cdot g\circ L$ where $H=h-S\cdot b\circ L$. This tells us that a fractal interpolation function f is uniquely defined by three functions H, S, and L, of the special forms defined above.

The fixed point f of T interpolates the data $\{(X_j,Y_j):j=0,1,2,...,N\}$ and is an example of a fractal interpolation function [1]. One way to evaluate f is to use

$$f=\lim_{k\to\infty}T^k(f_0),$$

where $f_0\in C^*$. The rate of convergence is governed by

$$\left\|f-T^k(f_0)\right\|_\infty \le s^k\left\|f-f_0\right\|_\infty.$$

4. The metric space $(I\times\mathbb{R},d_q)$

The following metric generalizes the "taxi-cab" metric. We will need it in the proof of Theorem 4.

Proposition 1. *Let $\alpha,\beta>0$ and $q:I\to\mathbb{R}$. Let $d_q:(I\times\mathbb{R})\times(I\times\mathbb{R})\to[0,\infty)$ be defined by*

$$d_q((x_1,y_1),(x_2,y_2)) := \alpha\left|x_1-x_2\right|+\beta\left|(y_1-q(x_1))-(y_2-q(x_2))\right|,$$

for all (x_1,y_1), $(x_2,y_2)\in I\times\mathbb{R}$. Then d_q is a metric on $I\times\mathbb{R}$. If q is continuous then $(I\times\mathbb{R},d_q)$ is a complete metric space.

Proof. Clearly $d_q((x_2,y_2),(x_1,y_1)) = d_q((x_1,y_1),(x_2,y_2))\ge 0$. Suppose that $d_q((x_1,y_1),(x_2,y_2))=0$. Then $\alpha\left|x_1-x_2\right|+\beta\left|(y_1-b(x_1))-(y_2-b(x_2))\right|=0$ which implies $x_1=x_2$. Hence $\left|(y_1-b(x_1))-(y_2-b(x_1))\right|=0$ which implies $y_1=y_2$.

Demonstration that d obeys the triangle inequality. Let $(x_i,y_i)\in I\times\mathbb{R}$, for $i=1,2,3$. Write $q_i=q(x_i)$ for $i=1,2,3$. We have

$$\begin{aligned} & d_q((x_1,y_1),(x_2,y_2))+d_q((x_2,y_2),(x_3,y_3)) \\ &= \alpha\left|x_1-x_2\right|+\beta\left|(y_1-q_1)-(y_2-q_2)\right|+\alpha\left|x_2-x_3\right|+\beta\left|(y_2-q_2)-(y_3-q_3)\right| \\ &= \alpha(\left|x_1-x_2\right|+\left|x_2-x_3\right|)+\beta(\left|(y_1-q_1)-(y_2-q_2)\right|+\left|(y_2-q_2)-(y_3-q_3)\right|) \\ &\ge \alpha(\left|x_1-x_3\right|)+\beta(\left|(y_1-q_1)-(y_2-q_2)\right|+\left|(y_2-q_2)-(y_3-q_3)\right|) \\ &\ge \alpha(\left|x_1-x_3\right|)+\beta(\left|(y_1-q_1)-(y_3-q_3)\right|)=d_q((x_1,y_1),(x_3,y_3)). \end{aligned}$$

To prove completeness in the case that q is continuous, let $\{(x_k,y_k)\}_{k=1}^\infty$ denote a Cauchy sequence with respect to the metric d_q. Given $\varepsilon>0$ we can find an integer $N(\varepsilon)$ so that

$$\alpha\left|x_k-x_l\right|+\beta\left|(y_k-q(x_k))-(y_l-q(x_l))\right|<\varepsilon$$

whenever $k,l>N(\varepsilon)$. It follows that $\{x_k\}$ is a Cauchy sequence with respect to the Euclidean norm, and so it converges, with limit $x^*\in I$. Since q is continuous,

it now follows that $\{q(x_k)\}$ converges to some limit $q^* \in \mathbb{R}$. In turn, it follows that $\{y_k\}$ converges to some $y^* \in \mathbb{R}$. Hence $\{(x_k, y_k)\}_{k=1}^{\infty}$ converges to $(x^*, y^*) \in I \times \mathbb{R}$. It follows that $(I \times \mathbb{R}, d)$ is complete. $\square$

5. Fractal interpolants as attractors of iterated function systems

Here we characterize the graph of the fixed point f of T as an attractor of an IFS. Define $w_n : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$w_n(x, y) := (l_n(x), h(l_n(x)) + S(l_n(x))(y - b(x))).$$

Define an IFS by

$$\mathcal{W} := (I \times \mathbb{R}; w_1, w_2, ..., w_N).$$

Here we make use of the metric d_q of Theorem 1 with $q = f$, the fixed point of T. Let $B \geq 0$ and let

$$\mathbb{X} := \{(x, y) : x \in I, |y - f(x)| \leq B\}.$$

It is readily verified that, when Theorem 3 holds, namely when $s < 1, \mathcal{W}(\mathbb{X}) \subset \mathbb{X}$. The following theorem gives conditions under which (i) the IFS $(\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive with respect to d_f and (ii) $\mathcal{W}$ has a unique attractor. This result is a substantial generalization of [12, Theorem 5.3, p. 140] which would require, in the present setting, that h is uniformly Lipschitz. Here, we avoid this restriction by using the metric d_q with $q = f$.

Theorem 4. *Let $s < 1$ and let $f \in C^{**}$ be the fixed point of T, as in Theorem 3. Let $l_n : I \to I$ have Lipschitz constant $\lambda_l < 1$, such that $|l_n(x_1) - l_n(x_2)| \leq \lambda_l |x_1 - x_2|$ for all $x_1, x_2 \in I$, for all n. Let $S : I \to [-s, s]$ have Lipschitz constant λ_S, so that $|S(x_1) - S(x_2)| \leq \lambda_S |x_1 - x_2|$ for all $x_1, x_2 \in I$. Then the IFS $(\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive with respect to the metric d_f with $\alpha = 1$ and $0 < \beta < (1 - \lambda_l)/\lambda_S B \lambda_l$. In particular, under these conditions, the IFS $\mathcal{W}$ has a unique attractor $A = \Gamma(f)$, the graph of f, with basin of attraction $I \times \mathbb{R}$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{X}$. We have

$$\begin{aligned}
&d_f(w_n(x_1, y_1), w_n(x_2, y_2)) - \alpha |l_n(x_1) - l_n(x_2)| \\
&= \beta |h(l_n(x_1)) + S(l_n(x_1))(y_1 - b(x_1)) - f(l_n(x_1)) \\
&- h(l_n(x_2)) + S(l_n(x_2))(y_2 - b(x_2)) - f(l_n(x_2))| \\
&= \beta |\,(S(l_n(x_1))(y_1 - f(x_1))) - (S(l_n(x_2))(y_2 - f(x_2)))\,| \\
&\leq \beta\, |S(l_n(x_1))| \cdot |(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&+ |S(l_n(x_1)) - S(l_n(x_2))| \cdot |(y_2 - f(x_2))| \\
&\leq \beta s |(y_1 - f(x_1)) - (y_2 - f(x_2))| + \beta \lambda_l \lambda_S B\, |x_1 - x_2|.
\end{aligned}$$

Hence

$$\begin{aligned}
&d_f(w_n(x_1, y_1), w_n(x_2, y_2)) \\
&\leq (\alpha \lambda_l + \beta \lambda_S \lambda_l B)\, |x_1 - x_2| + \beta s |(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&\leq (\alpha + \beta \lambda_S B)\, \lambda_l\, |x_1 - x_2| + \beta s |(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&\leq c \cdot (\alpha\, |x_1 - x_2| + \beta\, |(y_1 - f(x_1)) - (y_2 - f(x_2))|)
\end{aligned}$$

where $c := \max\{s, \lambda_l + \beta \lambda_S B \lambda_l / \alpha\}$. Since $\lambda_l < 1$ we can choose $\alpha, \beta > 0$ so that $c < 1$. For example, we can choose $\alpha = 1$ and $0 < \beta < (1 - \lambda_l)/\lambda_S B \lambda_l$.

It follows that the IFS $\widetilde{\mathcal{W}} := (\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive, and hence it has a unique attractor. This attractor must be $\Gamma(f)$ because a contractive IFS has a unique nonempty compact invariant set and it is readily verified that $\widetilde{\mathcal{W}}(\Gamma(f)) = \Gamma(f)$. Since we can choose the constant B arbitrarily large, it now follows that $\mathcal{W}$ has a unique attractor, namely $\Gamma(f)$. (But we have not provided a metric with respect to which $\mathcal{W}$ is contractive!) □

Note that
$$\Gamma(T(g)) = \mathcal{W}\left(\Gamma(g)\right), \text{ for all } g \in C.$$
When, for example, S is Lipschitz, $s < 1$, and the functions l_n are contractive, the graph of the fractal interpolant f can be approximated by the "chaos game" algorithm. (See [2] and [4] for new topological viewpoints of the "chaos game.")

6. Bilinear fractal interpolation

We consider a specific example of the preceding theory. Let
$$l_n(x) := X_{n-1} + \left(\frac{X_n - X_{n-1}}{X_N - X_0}\right)(x - X_0)$$
and let
$$S(x) := s_n(l_n^{-1}(x))$$
for $x \in [X_{n-1}, X_n]$, $n = 1, \ldots, N$, where
$$s_n(x) := s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_n - X_{n-1}}\right)(x - X_{n-1}),$$
with $\{s_j : j = 0, 1, 2, ..., N\} \subset (-1, 1)$. Then S is continuous and
$$\begin{aligned} |S(x)| &\leq \max\{|s_n(l_n^{-1}(x))| : x \in [X_{n-1}, X_n], n \in \{1, 2, ..., N\}\} \\ &= \max\{|s_j| : j = 0, 1, ..., N\} =: s < 1. \end{aligned}$$
Let
$$b(x) := Y_0 + \left(\frac{Y_N - Y_0}{X_N - X_0}\right)(x - X_0)$$
and let
$$h(x) := Y_{n-1} + \left(\frac{Y_n - Y_{n-1}}{X_n - X_{n-1}}\right)(x - X_{n-1}).$$
Theorem 3 implies that T has a unique fixed point f. Specifically, f is the unique solution of the set of functional equations
$$f(l_n(x)) - h(l_n(x)) = [s_{n-1} + (s_n - s_{n-1})x][f(x) - b(x)], \; x \in I.$$
We refer to f as a *bilinear fractal interpolant*. The reason for this name is that in this case the functions w_n of the IFS $\mathcal{W}$ take the form
$$w_n(x, y) := (l_n(x), a + bx + cy + dxy),$$
where a, b, c, d are real constants. Functions of the form $(x, y) \mapsto a + bx + cy + dxy$ are called *bilinear* in the computer graphics literature.

Specifically,
$$\begin{aligned} w_n(x, y) = (l_n(x), Y_{n-1} &+ \left(\frac{Y_n - Y_{n-1}}{X_N - X_0}\right)(x - X_0) \\ &+ \left[s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_N - X_0}\right)(x - X_0)\right]\left[y - Y_0 - \left(\frac{Y_N - Y_0}{X_N - X_0}\right)(x - X_0)\right]. \end{aligned} \tag{6.1}$$

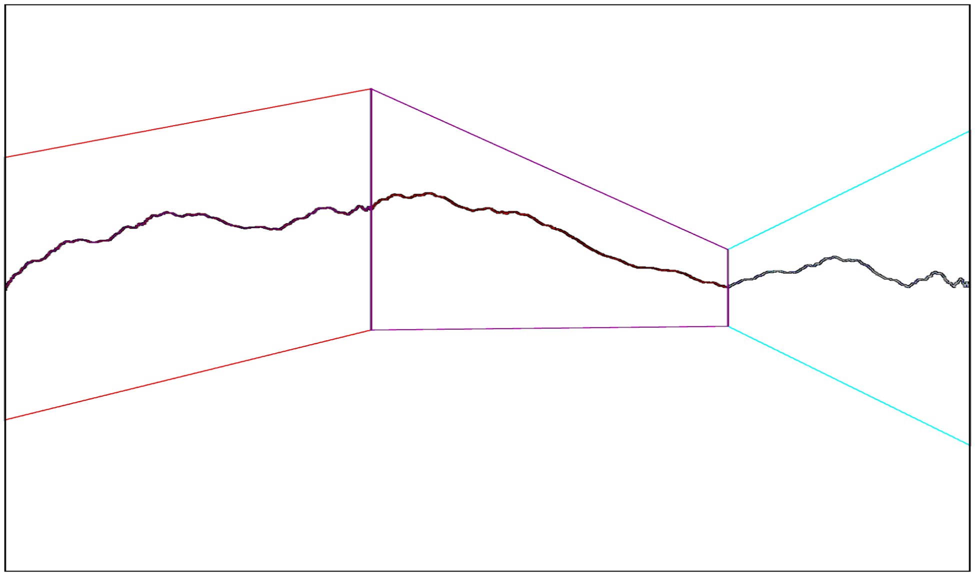

FIGURE 1. A fractal interpolation function defined by three bilinear transformations. See text.

In particular note that

$$w_n(X_N, y) = (X_n, Y_n + s_n(y - Y_N)) \text{ and } w_{n+1}(X_0, y) = (X_n, Y_n + s_n(y - Y_0)).$$

It follows that the images of any (possibly degenerate) parallelogram with vertices at $(X_0, Y_0 \pm H)$ and $(X_N, Y_N \pm H)$, for $H \in \mathbb{R}$ under the IFS fit together neatly, as illustrated in Figure 1.

7. Box dimension of bilinear interpolants

For present purposes we define the box-counting or box dimension of a bounded set $M \subset \mathbb{R}^n$ to be

$$\dim_B M := \lim_{\varepsilon \to 0+} \frac{\log \mathcal{N}_\varepsilon(M)}{\log \varepsilon^{-1}}, \tag{7.1}$$

where $\mathcal{N}_\varepsilon(M)$ is the minumum number of square boxes, with sides parallel to the axes, whose union contains M. By the statement "$\dim_B M = D$" we mean that the limit in equation (7.1) exists and equals D.

Remark 1. *It suffices to compute the box dimension of M using only discrete values of ε, such as $\varepsilon := N^{-m}$, for any $1 < N \in \mathbb{N}$ and $m \in \mathbb{N}$.*

In the case where M is the graph $\Gamma(f)$ of a function f, knowledge of the box dimension of $\Gamma(f)$ provides information about the smoothness of f since $\dim_B \Gamma(f)$ is related to Hölder exponents associated with f. (See, for example, [13, Section 12.5].)

The next result gives an explicit formula for the box dimension of the graph of a bilinear fractal interpolant. The proof is based on arguments first applied in [7].

Theorem 5. *Let $\mathcal{W}$ denote the bilinear IFS defined above, and let $\Gamma(f)$ denote its attractor. Let $a_n = 1/N$ for $n = 1, 2, ..., N$, and let $\sum_{n=1}^{N} \frac{s_{n-1}+s_n}{2} > 1$. If $\Gamma(f)$ is not a straight line segment then*

$$\dim_B \Gamma(f) = 1 + \frac{\log \sum_{n=1}^{N} \dfrac{s_{n-1}+s_n}{2}}{\log N};$$

otherwise $\dim_B \Gamma(f) = 1$.

Proof. Note that in the computation of the box dimension of $\Gamma(f)$ it suffices to consider covers of $\Gamma(f)$ whose elements are squares of side N^{-r}, $r \in \mathbb{N}_0$. Denote by $\mathcal{C}_0(r)$ a cover of $\Gamma(f)$ consisting of a finite number of squares of side N^{-r}, $r \in \mathbb{N}_0$. Now consider a specific cover $\mathcal{C}(r)$ of $\Gamma(f)$ of the form

$$\mathcal{C}(r) := \left\{ \left[\frac{k-1}{N^r}, \frac{k}{N^r}\right] \times \left[a, a + \frac{1}{N^r}\right] : r \in \mathbb{N}_0;\ k = 1, \ldots, N^r;\ a \in \mathbb{R} \right\}. \tag{7.2}$$

By the compactness of $\Gamma(f)$, there exists a minimal cover $\mathcal{C}_0^*(r)$ of $\Gamma(f)$ and also a minimal cover $\mathcal{C}^*(r)$ of $\Gamma(f)$ of the form (7.2). Denote by $\mathcal{N}_0(r)$, respectively, $\mathcal{N}(r)$ the cardinality of these minimal covers. Since covers of the form (7.2) are more restrictive, we have $\mathcal{N}_0(r) \leq \mathcal{N}(r)$. On the other hand, every $(N^{-r} \times N^{-r})$-square in $\mathcal{C}_0^*(r)$ can be covered by at most two $(N^{-r} \times N^{-r})$-squares from a cover of the form (7.2). Thus, $\mathcal{N}(r) \leq 2\mathcal{N}_0(r)$. Hence, when computing the box dimension of $\Gamma(f)$ it suffices to consider covers of the form (7.2).

By an affine transformation, one can always achieve that $y_0 = y_N = 0$. Since the box dimension is invariant under an affine transformation, we will restrict ourselves to this particular case.

To this end, let $r \in \mathbb{N}_0$ be fixed. Let $\mathcal{C}(r)$ be a minimal cover of $\Gamma(f)$ of cardinality $\mathcal{N}(r)$ consisting of squares of side N^{-r} whose interiors are disjoint. Let $\mathcal{C}(r,k)$ be the collection of all squares in $\mathcal{C}(r)$ that lie between $x = \frac{k-1}{N^r}$ and $x = \frac{k}{N^r}$, $k = 1, \ldots, N^r$. Denote by $\mathcal{N}(r,k)$ the cardinality of $\mathcal{C}(r,k)$, and let

$$\mathcal{R}(r,k) := \bigcup_{C_i \in \mathcal{C}(r,k)} C_i.$$

As $\mathcal{C}(r)$ is a cover of $\Gamma(f)$ of minimal cardinality, every square in $\mathcal{C}(r)$ must meet $\Gamma(f)$, and since f is continuous on I, the set $\mathcal{R}(r,k)$ must be a rectangle of width N^{-r} and height $h(r) := N^{-r}\mathcal{N}(r,k)$. Note that $\mathcal{N}(r) = \sum_{k=1}^{N^r} \mathcal{N}(r,k)$.

Now apply the mappings w_i, $i = 1, \ldots, N$, defined in (6.1) to the rectangle $\mathcal{R}(r,k)$. The image of $\mathcal{R}(r,k)$ under w_i is a trapezoid contained in the strip $\left[\dfrac{l(k,i)-1}{N^{r+1}}, \dfrac{l(k,i)}{N^{r+1}}\right] \times \mathbb{R}$, with $l(k,i) := k + (i-1)N^r$. Observe that

$$\mathcal{N}(r+1) = \sum_{i=1}^{N} \sum_{k=1}^{N^r} \mathcal{N}(r+1, l(k,i)).$$

The fixed point equation for $\Gamma(f)$, namely, $\Gamma(f) = \bigcup\limits_{i=1}^{N} w_i(\Gamma(f))$, implies that

$$\Gamma(f) \subseteq \bigcup_{i=1}^{N} w_i \left(\bigcup_{k=1}^{N^r} \mathcal{R}(r,k) \right).$$

Now, each trapezoid $w_i(\mathcal{R}(r,k))$ is contained in a rectangle of width $N^{-(r+1)}$ and height

$$\frac{|a_i|}{N^r} + \mu_{ik}\, h(r),$$

where $\mu_{ik} := \max\{s_{i-1}+(s_i-s_{i-1})\frac{k-1}{N^r}, s_{i-1}+(s_i-s_{i-1})\frac{k}{N^r}\}$. Note that $s_{i-1}+(s_i-s_{i-1})\frac{k-1}{N^r} > 0$ and $s_{i-1}+(s_i-s_{i-1})\frac{k}{N^r} > 0$. We write $\mu_{ik} = s_{i-1}+(s_i-s_{i-1})\frac{\varepsilon(k)}{N^r}$, with $\varepsilon(k)$ being either $k-1$ or k. Therefore,

$$\mathcal{N}(r+1, l(k,i)) \le \Big(\frac{a_i}{N^r} + \mu_{ik}\, h(r)\Big)\, N^{r+1} + 2 = N\mu_{ik}\,\mathcal{N}(r,k) + (N|a_i| + 2).$$

Summation over $k = 1,\dots,N^r$ and $i = 1\dots,N$ produces

$$\mathcal{N}(r+1) \le N\left[\left(\sum_{i=1}^{N} s_{i-1}\right)\mathcal{N}(r) + \frac{s_N - s_0}{N^r}\sum_{k=1}^{N^r}\varepsilon(k)\mathcal{N}(r,k)\right] + c_1\, N^{r+1},$$

where we set $c_1 := \sum_{i=1}^N |a_i| + 2$.

To estimate the sum over k, we employ Corollary 3.3 in [6] with $p_k := \frac{\mathcal{N}(r,k)}{\mathcal{N}(r)}$ and $m := 1$. This yields

$$\begin{aligned}\sum_{k=1}^{N^r}\varepsilon(k)\mathcal{N}(r,k) \le \sum_{k=1}^{N^r} k\,\mathcal{N}(r,k) &\le \frac{\mathcal{N}(r)}{2}\left(\left[\sum_{k=1}^{N^r} p_k^{1/2}\right]^2 + 1\right)\\ &\le \frac{\mathcal{N}(r)}{2}\left(\left[\sum_{k=1}^{N^r} p_k\right]N^r + 1\right) = \frac{\mathcal{N}(r)}{2}\,(N^r+1)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{N}(r+1) &\le N\left(\gamma + \frac{|s_N - s_0|}{2N^r}\right)\mathcal{N}(r) + c_1 N^{r+1}\\ &= (N\gamma)[1+\beta N^{-r}]\,\mathcal{N}(r) + c_1 N^{r+1},\end{aligned}$$

where we set $\gamma := \sum_{n=1}^N \frac{s_{n-1}+s_n}{2}$ and $\beta := \frac{|s_N - s_0|}{2\gamma}$. Note that $\gamma > 0$. Induction on r finally yields

$$\begin{aligned}\mathcal{N}(r) &\le (N\gamma)^r \prod_{j=0}^{r-1}[1+\beta N^{-(r-j)}]\,\mathcal{N}(1)\\ &\qquad + c_1 N^r\left(1 + \gamma[1+\beta N^{-r}] + \cdots + \gamma^{r-1}\prod_{j=0}^{r-1}[1+\beta N^{-(r-j)}]\right)\\ &= (N\gamma)^r \prod_{l=0}^{r}[1+\beta N^{-l}]\,\mathcal{N}(1) + c_1 N^r\left(1 + \sum_{j=1}^{r-1}\gamma^j \prod_{l=r-1-k}^{r}[1+\beta N^{-l}]\right).\end{aligned}$$

Next, we show that the two products in the above expression converge absolutely as $r\to\infty$. We consider the first product, the argument for the second one is similar. Note that $\{\prod_{l=0}^r[1+\beta N^{-l}] \,:\, r\in\mathbb{N}\}$ converges absolutely iff $\{\sum_{l=0}^r \beta N^{-l} \,:\, r\in\mathbb{N}\}$ converges absolutely. The latter sum, however, converges to $N\beta/(N-1)$. Denote the value of $\prod_{l=0}^\infty[1+\beta N^{-l}]$ by $P(\beta)$ and observe that $P(\beta) \le \exp(N\beta/(N-1))$.

Depending on the value of γ, two cases need to be considered.

Case I: $\gamma \le 1$**.** This implies that $\mathcal{N}(r) \le N^r\,[P(\beta)\mathcal{N}(1) + c_1(1+(r-1)P(\beta))]$. Hence,

$$\dim_B \Gamma(f) \le \lim_{r\to\infty} \frac{\log(N^r\,[P(\beta)\mathcal{N}(1) + c_1(1+(r-1)P(\beta))])}{\log N^r} = 1.$$

Case II: $\gamma > 1$**.** Here, we obtain

$$\begin{aligned}\mathcal{N}(r) &\le (\gamma N)^r\, P(\beta)\,\mathcal{N}(1) + c_1(\gamma N)^r \left(\gamma^{-r} + P(\beta)\sum_{j=1}^{r-1}\gamma^{j-r}\right)\\ &\le (\gamma N)^r\left[P(\beta)\mathcal{N}(1) + c_1 \max\{1, P(\beta)\}(1-\gamma)^{-1}\right] =: c_2(\gamma N)^r.\end{aligned}$$

Thus,

$$\dim_B \Gamma(f) \le \lim_{r\to\infty} \frac{\log c_2\,(\gamma N)^r}{\log N^r} = 1 + \frac{\log\gamma}{\log N}.$$

To find a non-trivial lower bound for $\dim_B \Gamma(f)$, note that since f is a continuous function, $\dim_B \Gamma(f) \ge 1$. If $\Gamma(f)$ is a line segment, i.e., if the set of data $\mathcal{J} := \{(X_j, Y_j \,:\, j = 0, 1, \ldots, N)\}$ is collinear, then $\Gamma(f) = [0,1]$. Hence, $\dim_B \Gamma(f) = 1$.

Suppose then that $\gamma > 1$ and that $\Gamma(f)$ is not a line segment, i.e., $\mathcal{J}$ is not collinear. Since each $C_i \in \mathcal{C}(r,k)$ meets $\Gamma(f)$, the image of C_i under the maps w_i, $i = 1, \ldots, N$, must also meet $\Gamma(f)$. Hence, $\mathcal{C}(r+1, l(i,k))$ must at least cover a rectangle of width $N^{-(r+1)}$ and height $(|\nu_{ik}|[\mathcal{N}(r,k) - 2] - |a_i|)\,N^{-r}$, where $\nu_{ik} := \min\{s_{i-1} + (s_i - s_{i-1})\frac{k-1}{N^r}, s_{i-1} + (s_i - s_{i-1})\frac{k}{N^r}\}$. This then implies that

$$\mathcal{N}(r+1, l(k,i)) \ge N\,(|\nu_{ik}|\,[\mathcal{N}(r,k) - 2] - |a_i|) - 2.$$

Summation over k and i, and using the rearrangement inequality [8, Equation 10.2.1] together with the fact that if the sequence $\{\mathcal{N}(r,k) \,:\, k = 1, \ldots, N^r\}$ is arranged in decreasing order,

$$\sum_{k=1}^{N^r} k\,\mathcal{N}(r,k) \ge \frac{N^r - 1}{2}\,\sum_{k=1}^{N^r}\mathcal{N}(r,k) = \frac{N^r-1}{2}\,\mathcal{N}(r),$$

gives

$$\mathcal{N}(r+1) \ge (\gamma\,N)\,\left[1 + \beta N^{-(r+1)}\right]\,\mathcal{N}(r) - c_3\,N^{r+1} \ge (\gamma\,N)\,\mathcal{N}(r) - c_3\,N^{r+1},$$

with $c_3 := 2 + N\displaystyle\sum_{i=1}^{N}(|a_i| + 2\min\{s_{i-1}, s_i\})$. Induction over r yields

$$\begin{aligned}\mathcal{N}(r+1) &\ge (\gamma N)^m \mathcal{N}(r-m+1) - c_3 N^{r+1}(1 + \gamma + \cdots + \gamma^{m-1})\\ &\ge (\gamma N)^{r-m+1}\mathcal{N}(m) - \frac{c_3}{1-\gamma^{-1}}\,N^{r+1}\\ &= (\gamma\,N)^{r-m+1}\,\left(\mathcal{N}(m) - \frac{c_3 N^m}{1-\gamma^{-1}}\right),\end{aligned}$$

for all $m \in \mathbb{N}$ with $1 \le m \le r$.

To complete the proof of the theorem, the following lemma is needed.

Lemma 3. *Suppose that f is a bilinear fractal interpolant. Denote by $\mathcal{N}(r)$ the cardinality of a minimal cover of $\Gamma(f)$ of the from (7.2). If $\gamma := \sum_{i=1}^N \frac{s_{i-1}+s_i}{2} > 1$ and if $\Gamma(f)$ is not a line segement then*

$$\lim_{r\to\infty} \frac{N^r}{\mathcal{N}(r)} = 0.$$

Proof. The assumption that $\Gamma(f)$ is not a line segement implies the existence of at least one index $i_0 \in \{1, \ldots, N-1\}$ so that

$$\delta := |y_{i_0}| > 0.$$

Since f is continuous on $[0,1]$, we have that $\mathcal{N}(r) \geq \delta N^r$, for any finite code σ of length $r \in \mathbb{N}$. Note that I is mapped to the line segments $\overline{(x_{i-1}, y_{i-1}), (x_i, y_i)}$, implying that

$$\mathcal{N}(r) \geq \sum_{i=1}^N [s_{i-1} + (s_i - s_{i-1})\, x_0]\ \delta\, N^r,$$

where we set $x_0 := i_0/N$. Inductively, this yields

$$\mathcal{N}(r) \geq \sum_{i_1,\cdots,i_k=1}^N \prod_{\ell=1}^k \left[s_{i_\ell -1} + (s_{i_\ell} - s_{i_\ell -1})\, L_{i_{\ell+1}\cdots i_k}(x_0)\right]\ \delta\, N^r,$$

with $L_{i_1\cdots i_m} := L_{i_1} \circ \cdots \circ L_{i_m}$ and $L_\emptyset(x_0) := x_0$. We note that

$$L_{i_1\cdots i_m}(x) = \frac{1}{N^m}\left(x + \sum_{n=1}^m N^{n-1}(i_n - 1)\right),$$

for all $m \in \mathbb{N}$ and $x \in [0,1]$. The rearrangement inequality [8, Equation 10.2.1] implies that for all $m \in \mathbb{N}$

$$\sum_{n=1}^m N^{n-1}(i_n - 1) \geq \sum_{n=1}^m N^{n-1}(i_n^* - 1),$$

where the $i_n^* \in \{1, \ldots, N\}$ are taken in decreasing arrangement. Evaluating the latter sum and simplifying, one obtains the existence of an $m_0 \in \mathbb{N}$ such that

$$L_{i_1\cdots i_m}(x_0) \geq 1, \quad \forall m \geq m_0.$$

Hence,

$$\mathcal{N}(r) \geq c\, \gamma^r\, \delta\, N^r, \quad \forall r \geq m \geq m_0.$$

and for some constant $c > 0$. As, by assumption $\gamma > 1$, the statement follows. □

Now, by Lemma 3, one can choose r and m large enough so that

$$\mathcal{N}(m) - \frac{c_3 N^m}{1-\gamma} > 0,$$

and, therefore, $\mathcal{N}(r) \geq c_4\, (\gamma\, N)^r$, for a constant $c_4 > 0$. Hence, $\dim_B \Gamma(f) \geq 1 + \dfrac{\log \gamma}{\log N}$. □

Remark 2. *The proof of Theorem 5 shows in particular that for a given word $\sigma = \sigma_1 \ldots \sigma_r$ of finite length $|\sigma|$, there exist constants $0 < \underline{c} \leq \overline{c}$ such that*

$$\underline{c}\, (\gamma\, N)^{|\sigma|} \leq \mathcal{N}(|\sigma|) \leq \overline{c}\, (\gamma\, N)^{|\sigma|}.$$

Moreover, if $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ *denotes the image of* $\Gamma(f)$ *under the maps* $w_{\sigma_1\cdots\sigma_r} := w_{\sigma_1}\circ\cdots\circ w_{\sigma_r}$ *over the subinterval* $L_{\sigma_1\cdots\sigma_r}(I)$*, then there also exist constants* $0 < \underline{c}^* \leq \overline{c}^*$ *such that*

$$\underline{c}^*\,\lambda_{\sigma_1}\cdots\lambda_{\sigma_r}\,N^{|\sigma|} \leq \mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|) \leq \overline{c}^*\,\lambda_{\sigma_1}\cdots\lambda_{\sigma_r}\,N^{|\sigma|}, \tag{7.3}$$

where $\mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|)$ *denotes the minimum number of* $N^{-|\sigma|}\times N^{-|\sigma|}$*-squares from a cover of the form (7.2) needed to cover* $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ *and* $\lambda_i := \frac{s_{i-1}+s_i}{2}$*,* $i = 1,\ldots,N$*.*

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